JOURNAL OF PURE AND APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 133 (1998) 289–316

The transfer map in topological Hochschild homology

Christian Schlichtkrull*

Department of Mathematics, Aarhus University, DK-8000 Aarhus C, Denmark

Communicated by C.A. Weibel; received 28 November 1996; received in revised form 19 March 1997

Abstract

We consider the topological Hochschild homology (THH) of a group ring $R[G]$, and calculate the restriction map (or transfer) associated with a subgroup $K \subset G$ of finite index in terms of ordinary group homology transfers. This gives information on the corresponding restriction map in Quillen's K-theory via the topological Dennis trace $\mathrm{tr}: \mathrm{K}(R[G]) \to \mathrm{THH}(R[G])$. More generally, we consider group rings for "rings up to homotopy" (FSP's) and calculate the THH-restriction map in terms of transfers in generalized homology theories. © 1998 Elsevier Science B.V. All rights reserved.

AMS Classification: Primary: 19D99, 55R12; secondary: 19D10

0. Introduction

One possible way to study Quillens algebraic K-theory $\mathrm{K}(R)$ of a ring R, is to consider its relationship with the topological Hochschild homology $\mathrm{THH}(R)$ as defined by Bökstedt, see [5] or [13]. The latter is a topological version of the ordinary Hochschild homology, and the topological Dennis trace is a natural map

$$\mathrm{tr}: \mathrm{K}(R) \to \mathrm{THH}(R).$$

This is a non-trivial invariant. By a theorem of Dundas and McCarthy [8], the stable K-theory of R is equivalent to $\mathrm{THH}(R)$.

For a discrete group G and a subgroup K of finite index, there is an inclusion of group rings $R[K] \to R[G]$, and a corresponding restriction map (or transfer)

$$\mathrm{Res}: \mathrm{K}(R[G]) \to \mathrm{K}(R[K]).$$

* E-mail: krull@mi.aau.dk.

PII: S0022-4049(97)00117-5

The construction is simple: The inclusion of rings $R[G] \to \mathrm{End}_{R[K]}(R[G])$ induced from the left multiplication of G on $R[G]$ gives a map on K-theory, and the restriction map is induced from this by Morita equivalence. (A choice of coset representatives provides an $R[K]$-basis for $R[G]$.) Though easy to define, the K-theoretical restriction map has proved hard to analyze effectively. However, there is a corresponding restriction map in topological Hochschild homology, and a commutative diagram

$$\begin{tikzcd} \mathrm{K}(R[G]) \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{Res}"'] & \mathrm{THH}(R[G]) \arrow[d, "\mathrm{Res}"] \\ K(R[K]) \arrow[r, "\mathrm{tr}"] & \mathrm{THH}(R[K]). \end{tikzcd} \tag{0.1}$$

In this paper, we describe completely the behavior of the THH-restriction map on homotopy groups in terms of the well-known transfers in ordinary group homology, cf. [7, 3.9]. Let $\langle G\rangle$ denote the conjugacy classes of G, and for $\omega \in \langle G\rangle$ write $C_G(\omega)$ to mean the centralizer of σ in G for some representative $\sigma \in \omega$. (This is independent of the choice of σ up to isomorphism.) We also write $\pi_i = \pi_i(\mathrm{THH}(R))$, and consider it as a trivial $C_G(\omega)$-module. With this notation we have

$$\pi_n(\mathrm{THH}(R[G])) = \bigoplus_{\omega\in\langle G\rangle} \bigoplus_{i=0}^{n} \mathrm{H}_i(C_G(\omega), \pi_{n-i})$$

and similarly,

$$\pi_n(\mathrm{THH}(R[K])) = \bigoplus_{\lambda\in\langle K\rangle} \bigoplus_{i=0}^{n} \mathrm{H}_i(C_K(\lambda), \pi_{n-i}).$$

This gives a corresponding decomposition $\mathrm{Res}_n = \bigoplus_{\omega,\lambda} \mathrm{Res}^{\lambda}_{\omega}$, where

$$\mathrm{Res}^{\lambda}_{\omega} = \bigoplus_{i=0}^{n} \mathrm{Res}^{\lambda}_{\omega,i} : \bigoplus_{i=0}^{n} \mathrm{H}_i(C_G(\omega), \pi_{n-i}) \to \bigoplus_{i=0}^{n} \mathrm{H}_i(C_K(\lambda), \pi_{n-i}).$$

Theorem A. *Let $\omega \in \langle G\rangle$ and $\lambda \in \langle K\rangle$.*

(i) *If $\lambda \not\subseteq \omega$ then* $\mathrm{Res}^{\lambda}_{\omega} = 0$.

(ii) *If $\lambda \subseteq \omega$ then for any $\kappa \in \lambda$ we may take $C_G(\omega) = C_G(\kappa)$ and $C_K(\lambda) = C_K(\kappa)$, and* $\mathrm{Res}^{\lambda}_{\omega}$ *is then the usual transfer in group homology corresponding to the inclusion $C_K(\kappa) \to C_G(\kappa)$.*

A similar description of the restriction map in ordinary Hochschild homology has been given by Bentzen and Madsen, cf. [4].

The homotopy groups of THH(R) are not known in general but, for example,

$$\pi_i(\mathrm{THH}(\mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{for } i=0, \\ 0 & \text{for } i=2j,\ j \geq 1, \\ \mathbb{Z}/j\mathbb{Z} & \text{for } i=2j-1,\ j \geq 1. \end{cases}$$

More generally, Lindenstrauss and Madsen have calculated the homotopy groups of THH(R) when R is the integers in a finite extension of the rationals, see [11].

In fact, we shall work in the more general context of "rings up to homotopy", or, in Bökstedts formulation, *functors with smash products* (FSPs). For L any FSP, one can define the algebraic K-theory K(L), cf. [6, 5.4]. This construction generalizes both Quillens K-theory for discrete rings, and Waldhausens A-theory of spaces. (In the latter case A(X) is obtained from the FSP associated with the monoid of Moore loops on X.) Similarly, the topological Hochschild homology is defined for every FSP L. We also have a notion of group rings in the context of FSP's, and as in the linear case we have restriction maps and a diagram like 0.1 with R replaced by L.

The main problem in analyzing the THH-restriction map comes from the fact that there is no trace map $\mathrm{tr}: \mathrm{THH}(M_nL) \to \mathrm{THH}(L)$ inducing Morita equivalence. The standard proof of Morita equivalence (by an argument originally due to Waldhausen [18, Section 6]) consists in producing a new space which maps to both THH(M_nL) and THH(L) by equivalences. However, the lack of an explicit map is inconvenient (to say at least) for calculational purposes. We shall remedy this by constructing a new model $\mathrm{THH}^+(L)$ of topological Hochschild homology, together with an explicit trace map equivalence

$$\mathrm{tr}: \mathrm{THH}^+(M_nL) \to \mathrm{THH}^+(L).$$

This map is formally very similar to the trace inducing Morita equivalence in ordinary Hochschild homology, cf. [12, 1.2.1]. We construct $\mathrm{THH}^+(L)$ by fusing Bökstedts model of THH(L) with the Γ^+ construction of Barratt and Eccles [3]. In this way we get a space equivalent to THH(L), but with an enriched combinatorial structure so as to make the construction of the trace map possible.

In formulating the results on the restriction map for general FSPs, it is convenient to work in the stable category of spectra. (In this paper a spectrum E is simply a sequence of spaces E_n together with maps $S^1 \wedge E_n \to E_{n+1}$). In fact, THH($L$) and $\mathrm{THH}^+(L)$ are the spaces in degree zero of spectra we denote by TH(L) and $\mathrm{TH}^+(L)$, respectively, and all the maps in 0.1 are maps of spectra. Recall (e.g. from [2]) that to any spectrum E there is an associated homology theory denoted E_*. The general procedure for constructing the transfer in E_*-theory of an n-sheeted covering $X \to A$ consists in first producing a transfer map of suspension spectra

$$\mathrm{trf}: \Sigma^\infty(A_+) \to \Sigma^\infty(X_+)$$

(this is described in detail in [1, Section 5]), and then smash this map with the spectrum E. On homotopy groups there results a map $E_*(A) \to E_*(X)$.

Returning to the restriction map, we prove in Propostion 4.3 the existens of a commutative diagram of spectra:

$$\begin{array}{ccc}
\mathrm{TH}^+(L[G]) & \xrightarrow{\qquad\mathrm{Res}\qquad} & \mathrm{TH}^+(L[K]) \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
\bigvee_{\omega\in\langle G\rangle} \mathrm{TH}(L)\wedge\Sigma^\infty(BC_G(\omega)_+) & \xrightarrow{\vee\mathrm{id}\wedge\widetilde{\mathrm{Res}}_\omega^\lambda} & \bigvee_{\lambda\in\langle K\rangle} \mathrm{TH}(L)\wedge\Sigma^\infty(BC_K(\lambda)_+),
\end{array}$$

with the vertical maps being equivalences. In this diagram

$$\widetilde{\mathrm{Res}}_\omega^\lambda : \Sigma^\infty(BC_G(\omega)_+) \to \Sigma^\infty(BC_K(\lambda)_+)$$

is defined using the Barratt–Eccles model of the suspension spectrum, cf. (4.5). The general result then reads as follows.

Theorem B. *Let $\omega\in\langle G\rangle$ and $\lambda\in\langle K\rangle$.*

(i) *If $\lambda\not\subseteq\omega$ then $\widetilde{\mathrm{Res}}_\omega^\lambda\simeq *$.*

(ii) *If $\lambda\subseteq\omega$ and $\kappa\in\lambda$ then $\widetilde{\mathrm{Res}}_\omega^\lambda$ is the transfer corresponding to the covering $BC_K(\kappa)\to BC_G(\kappa)$ (take $BC_K(\kappa)=EC_G(\kappa)/C_K(\kappa)$).*

We prove this in Proposition 6.2. In fact, Theorem A follows from Theorem B, since in the special case where L is a discrete ring R, $\mathrm{TH}(R)$ is a generalized Eilenberg–MacLane spectrum. Therefore $\mathrm{TH}(R)_*$-theory reduces to ordinary homology with coefficients in the homotopy groups of $\mathrm{TH}(R)$.

For simplicity we work non-equivariantly in this paper, but in fact the maps involved are all cyclic maps. Therefore, the ideas presented here can also be used to study the restriction map on the fixed-points corresponding to the action of a finite cyclic group, and in this way we get information on the restriction map in TC. This program will be carried out in [16].

Finally, a short comment on notation. We have chosen the term *restriction map* for the "wrong way" map in topological Hochschild homology. The natural map $\mathrm{TH}(L[K])\to\mathrm{TH}(L[G])$ induced from the inclusion $K\subseteq G$ should then be called the *induction map*. This terminology is in accordance with the usual definitions in K-theory and group homology, but it differs from that of [13]. Thus, the "induction map" appearing in [13, 5.1.14] corresponds to our restriction map.

1. Pointed monoids and traces

In this paper we shall work in the category of pointed simplicial sets $\mathbf{Simp}_*$. This seems to be the most natural choice, since the mathematics involved will generally be of a combinatorial nature. However, the constructions are all very natural, and if the reader prefers to do so he can interpret everything in terms of topological spaces. In fact, we shall allow ourselves to use topological language such as spaces and subspaces even though we work simplicially. We say that a map $f : X \to Y$ of simplicial sets is a homotopy equivalence if this is the case for the topological realization, and similarly for homology.

As motivation for the construction of $\mathrm{THH}^{\mathrm{I}}$ in the next section, let us first see how to define the trace of a matrix with entries in a pointed monoid, assuming that at most one entry in each column is different from the basepoint.

Definition 1.1. A pointed monoid $(\Pi, \mathbf{1}, *)$ is a pointed set Π with a pointed associative multiplication $\mu : \Pi \wedge \Pi \to \Pi$ and a two-sided unit $\mathbf{1} \in \Pi$.

Example 1.2. (1) If M is an ordinary monoid one gets a pointed monoid M_+ by adding a disjoint basepoint.

(2) Let R be a ring. Then by forgetting the additive structure R becomes a pointed monoid with basepoint 0.

Given a pointed monoid Π, we construct the cyclic nerve $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)$ in analogy with the construction of the Hochschild complex, with smash substituted for tensor products. (For the general theory of cyclic sets, see eg. [12, Ch. 7].)

$$\begin{aligned}
&\mathrm{N}^{\mathrm{cy}}_{\wedge i}(\Pi) = \Pi^{\wedge(i+1)},\\
&d_v : \mathrm{N}^{\mathrm{cy}}_{\wedge i}(\Pi) \to \mathrm{N}^{\mathrm{cy}}_{\wedge(i-1)}(\Pi), \quad 0 \le v \le i,\\
&d_v(x_0, \ldots, x_i) = (\ldots, \mu(x_v, x_{v+1}), \ldots), \quad 0 \le v \le i-1,\\
&d_i(x_0, \ldots, x_i) = (\mu(x_i, x_0), \ldots, x_{i-1}),\\
&s_v : \mathrm{N}^{\mathrm{cy}}_{\wedge i}(\Pi) \to \mathrm{N}^{\mathrm{cy}}_{\wedge(i+1)}(\Pi), \quad 0 \le v \le i,\\
&s_v(x_0, \ldots, x_i) = (x_0, \ldots, x_v, \mathbf{1}, x_{v+1}, \ldots, x_i),\\
&t_i : \mathrm{N}^{\mathrm{cy}}_{\wedge i}(\Pi) \to \mathrm{N}^{\mathrm{cy}}_{\wedge i}(\Pi),\\
&t_i(x_0, \ldots, x_i) = (x_i, x_0, \ldots, x_{i-1}).
\end{aligned} \tag{1.1}$$

Let $M_n(\Pi)$ denote the multiplicative monoid of $n \times n$ matrices with entries in Π, such that each column has at most one entry different from the basepoint, or in other words

$$M_n(\Pi) = \mathrm{Map}_*([n], [n] \wedge \Pi), \quad [n] = \{0, \ldots, n\}.$$

This is again a pointed monoid and we may thus consider $\mathrm{N}^{\mathrm{cy}}_{\wedge}(M_n(\Pi))$. We want to define trace maps in this situation, analogous to the linear case of Hochschild homology, where we have a trace map

$$\begin{aligned}
&\mathrm{tr} : Z(M_n(R)) \to Z(R),\\
&\mathrm{tr}(A^0 \otimes \cdots \otimes A^i) = \sum_{j_0, \ldots, j_i} a^0_{j_i, j_0} \otimes \cdots \otimes a^i_{j_{i-1}, j_i}, \qquad A^v = (a^v_{st}).
\end{aligned} \tag{1.2}$$

Here $Z(R)$ denotes the Hochschild complex associated with the ring R, i.e. $Z_i(R) = R^{\otimes(i+1)}$ with the usual cyclic structure maps, cf. (1.1). In the case of a pointed monoid Π we shall define

$$\mathrm{tr} : \mathrm{N}^{\mathrm{cy}}_{\wedge}(M_n(\Pi)) \to \Gamma^+(\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)), \tag{1.3}$$

where Γ^+ is the Barratt–Eccles functor (which for connected X gives a simplicial model for $\Omega^\infty\Sigma^\infty(X)$).

We first explain tr in simplicial degree zero. Given $(a_{ij}) \in M_n(\Pi)$ we consider the string of elements

$$(a_{j(1)j(1)}, \dots, a_{j(m)j(m)}) \in \Pi^m$$

where $1 \le j(1) < \cdots < j(m) \le n$ and $a_{j(s)j(s)} \ne *$.

In simplicial degree one, we have $((a^0_{ij}), (a^1_{ij})) \in M_n(\Pi) \wedge M_n(\Pi)$. In view of (1.2) it is natural to consider the set

$$S = \{(a^0_{j_1 j_0}, a^1_{j_0 j_1}) \in \Pi \wedge \Pi : a^0_{j_1 j_0} \ne *, a^1_{j_0 j_1} \ne *\}.$$

To each j_0 there is at most one j_1 with $(a^0_{j_1 j_0}, a^1_{j_0 j_1}) \in S$ and similarly for each j_1 at most one j_0 with $(a^0_{j_1 j_0}, a^1_{j_0 j_1}) \in S$. By using either the natural ordering of the j_0's or the j_1's we thus get two different ways of ordering the elements in S, and we must take both orderings into account in order to get a cyclic map. To do this we proceed as follows. First, we choose an arbitrary ordering of the elements in S:

$$S = \{(a^0_{j_1(1)j_0(1)}, a^1_{j_0(1)j_1(1)}), \dots, (a^0_{j_1(m)j_0(m)}, a^1_{j_0(m)j_1(m)})\}.$$

Let Σ_m be the group of permutations of the set $\mathbf{m} = \{1, \dots, m\}$, and let $\alpha_0, \alpha_1 \in \Sigma_m$ be determined by the order of the $j_0(s)$'s and $j_1(s)$'s, respectively,

$$j_0(\alpha_0^{-1}(1)) < \cdots < j_0(\alpha_0^{-1}(m)), \qquad j_1(\alpha_1^{-1}(1)) < \cdots < j_1(\alpha_1^{-1}(m)).$$

Σ_m acts from the right on each coordinate in Σ_m^2 and from the right on $(\Pi \wedge \Pi)^m$ by permuting the coordinates. We then define

$$\begin{aligned}\mathrm{tr}((a^0_{ij}), (a^1_{ij})) &= [(\alpha_0, \alpha_1), (a^0_{j_1(1)j_0(1)}, a^1_{j_0(1)j_1(1)}), \dots, (a^0_{j_1(m)j_0(m)}, a^1_{j_0(m)j_1(m)})] \\ &\in \Sigma_m^2 \times_{\Sigma_m} (\Pi \wedge \Pi)^m.\end{aligned}$$

One can check that this is independent of the ordering of S.

Let us now recall the definition of the functor Γ^+ due to Barratt and Eccles [3]. We let $\mathbf{n} = \{1, \dots, n\}$ and write $\mathcal{M}(\mathbf{m}, \mathbf{n})$ for the set of all strictly increasing maps from $\mathbf{m}$ to $\mathbf{n}$. For $\sigma \in \Sigma_n$ and $\alpha \in \mathcal{M}(\mathbf{m}, \mathbf{n})$ the composite $\sigma\alpha$ is not necessarily strictly increasing, but there is a unique morphism $\sigma_*(\alpha) \in \mathcal{M}(\mathbf{m}, \mathbf{n})$ such that $\sigma_*(\alpha)(\mathbf{m}) = \sigma\alpha(\mathbf{m}) \subseteq \mathbf{n}$.

Definition 1.3. For $\alpha \in \mathcal{M}(\mathbf{m}, \mathbf{n})$ define the restriction map $\alpha^* : \Sigma_n \to \Sigma_m$ by commutativity of the diagram:

$$\begin{tikzcd} \mathbf{m} \arrow[r, "\alpha"] \arrow[d, "\alpha^*(\sigma)"'] & \mathbf{n} \arrow[d, "\sigma"] \\ \mathbf{m} \arrow[r, "\sigma_*(\alpha)"] & \mathbf{n}. \end{tikzcd}$$

It is easy to see that $(\alpha\beta)^* = \beta^*\alpha^*$ for $\alpha \in \mathscr{M}(\mathbf{m},\mathbf{n})$ and $\beta \in \mathscr{M}(\mathbf{l},\mathbf{m})$.

Definition 1.4. For a based simplicial set X there is a right action of Σ_n on X^n given by

$$(x_1,\ldots,x_n)\sigma = (x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

In the same manner a morphism $\alpha \in \mathscr{M}(\mathbf{m},\mathbf{n})$ induces a map $\alpha^* : X^n \to X^m$ by letting

$$\alpha^*(x_1,\ldots,x_n) = (x_{\alpha(1)},\ldots,x_{\alpha(m)}).$$

Given $\mathbf{x} \in X^n$ we say that α is *entire* for $\mathbf{x}$ if α^* only misses the basepoint, i.e. $i \notin \alpha(\mathbf{m})$ implies $x_i = *$.

Let E be the functor from sets to cyclic sets given by

$$\mathrm{E}X\colon [i] \mapsto \mathrm{Map}([i],X) = X^{i+1}, \tag{1.4}$$

for any set X. Explicitly, we have the simplicial structure maps

$$d_v(x_0,\ldots,x_i) = (\ldots,\hat{x}_v,\ldots),$$

$$s_v(x_0,\ldots,x_i) = (\ldots,x_v,x_v,\ldots)$$

and the cyclic operators

$$t_i(x_0,\ldots,x_i) = (x_i,x_0,\ldots,x_{i-1}).$$

This is a contractible simplicial set for all X. In particular, E applies to a discrete group G and gives a space EG with a free right G-action defined by component wise multiplication. Notice also that the restriction map (Definition 1.3) extends to a cyclic map $\alpha^* : \mathrm{E}\Sigma_n \to \mathrm{E}\Sigma_m$, using the functoriality of E.

For X a pointed simplicial set we have the bisimplicial set

$$\mathscr{U}(X) = \coprod_{m \ge 0} \mathrm{E}\Sigma_m \times X^m, \tag{1.5}$$

where X^m denotes the diagonal simplicial set in the multisimplicial cartesian product of X with itself m times. Consider the following relations on $\mathscr{U}(X)$:

(a) $(\mathbf{c},\mathbf{x}) \sim (\mathbf{c}\sigma,\mathbf{x}\sigma)$ for $\mathbf{c} \in \mathrm{E}\Sigma_m$, $\mathbf{x} \in X^m$ and $\sigma \in \Sigma_m$

(b) $(\mathbf{c},\mathbf{x}) \sim (\alpha^*\mathbf{c},\alpha^*\mathbf{x})$ for $\mathbf{c} \in \mathrm{E}\Sigma_m$, $\mathbf{x} \in X^m$ and $\alpha \in \mathscr{M}(\mathbf{m},\mathbf{n})$ entire for $\mathbf{x}$.

Definition 1.5 (*Barratt and Eccles* [3]). The bisimplicial set $\Gamma^+(X)$ has (i,j) simplices

$$\Gamma_i^+ X_j = \coprod_{m \geq 0} \mathrm{E}_i \Sigma_m \times X_j^m / \sim,$$

where $\sim$ is the equivalence relation generated by (a) and (b).

The elements in $\Gamma^+(X)$ are denoted $[\mathbf{c};\mathbf{x}]$ for $\mathbf{c} \in \Sigma_m^{i+1}$ and $\mathbf{x} \in X_j^m$. In the following, we shall often consider $\Gamma^+(X)$ as a simplicial set by restricting to the diagonal.

Lemma 1.6. *Γ^+ maps cyclic sets to cyclic sets.*

Proof. For a cyclic set X we give $\mathrm{E}\Sigma_n \times X^n$, the obvious structure of a bicyclic set. This structure is compatible with (a) and (b) in Definition 1.5, and by restriction to the diagonal we thus get an endofunctor Γ^+ on the category of cyclic sets. □

We are now ready to define the trace map (1.3). Assume that we are given an element $(A^0,\dots,A^i) \in \mathrm{N}_{\wedge i}^{\mathrm{cy}}(M_n(\Pi))$. Let

$$S = \{(j_0,\dots,j_i) \in \mathbf{n}^{i+1} : a^0_{j_i,j_0} \neq *,\dots,a^i_{j_{i-1},j_i} \neq *\}$$

and assume that S has cardinality m. We choose an ordering of S, that is a bijective map $\rho = (\rho_0,\dots,\rho_i) : \mathbf{m} \to S$. The maps ρ_v are all injective, and for $v = 0,\dots,i$ we let $\alpha_v^{-1} \in \Sigma_m$ be the ordering of $\mathbf{m}$ induced from the inclusion in $\mathbf{n}$ by ρ_v:

$$\rho_v(\alpha_v^{-1}(1)) < \cdots < \rho_v(\alpha_v^{-1}(m)).$$

Then, we define

$$\mathrm{tr}(A^0,\dots,A^i) = [\boldsymbol{\alpha}; \mathbf{a}(1),\dots,\mathbf{a}(m)] \in \Gamma_i^+(\mathrm{N}_{\wedge i}^{\mathrm{cy}}(\Pi)), \tag{1.6}$$

where $\boldsymbol{\alpha} = (\alpha_0,\dots,\alpha_i)$ and $\mathbf{a}(v) = (a^0_{\rho_i(v)\rho_0(v)},\dots,a^i_{\rho_{i-1}(v)\rho_i(v)})$.

The crucial observation is that $\mathrm{tr}(A^0,\dots,A^i)$ is independent of the ordering of S, and given this it is easy to prove the following.

Proposition 1.7. *The trace map*

$$\mathrm{tr} : \mathrm{N}_{\wedge}^{\mathrm{cy}}(M_n(\Pi)) \to \Gamma^+(\mathrm{N}_{\wedge}^{\mathrm{cy}}(\Pi))$$

is a map of cyclic sets.

There are inclusions $M_n(\Pi) \to M_{n+1}(\Pi)$ obtained by adding a $(n+1)$'s row and column consisting of basepoints alone. The trace is compatible with these inclusions, and we get an induced map

$$\mathrm{tr} : \mathrm{N}_{\wedge}^{\mathrm{cy}}(M_\infty(\Pi)) \to \Gamma^+(\mathrm{N}_{\wedge}^{\mathrm{cy}}(\Pi)),$$

where $M_\infty(\Pi) = \varinjlim M_n(\Pi)$.

2. The topological Hochschild spectrum

We first recall the definition of THH as given by Marcel Bökstedt. For further details see [5] or [13]. As in the original paper by Bökstedt we shall work in the simplicial category. Thus, a *functor with smash product* (an FSP) L is a functor from the category of pointed simplicial sets to itself together with natural transformations

$$\begin{aligned} &\mathbf{1}: X \to L(X) \quad \text{the unit,} \\ &\mu: L(X) \wedge L(Y) \to L(X \wedge Y) \quad \text{the multiplication.} \end{aligned} \tag{2.1}$$

These are supposed to satisfy the obvious associativity and unital conditions and to be compatible with the stabilization map

$$\sigma_{X,Y}: X \wedge L(Y) \to L(X \wedge Y).$$

Furthermore, we shall always assume an FSP to be connected, in the sense that there exists a constant c such that the maps

$$S^1 \wedge L(S^n) \to L(S^{n+1})$$

are $(2n-c)$-connected.

Example 2.1. Let G be a discrete group. The group FSP $\widetilde{G}$ is given on objects by $\widetilde{G}(X) = X \wedge G_+$ and has structure maps

$$\mathbf{1}: X \to X \wedge G_+, \qquad \mathbf{1}(x) = (x, 1),$$

$$\mu: X \wedge G_+ \wedge Y \wedge G_+ \to X \wedge Y \wedge (G \times G)_+ \xrightarrow{\mathrm{id} \wedge \mathrm{m}} X \wedge Y \wedge G_+,$$

where m is multiplication in G.

Example 2.2. The matrix FSP M_n has $M_n(X) = \mathrm{Map}_*([n], [n] \wedge X)$ where $[n] = \{0, \ldots, n\}$ is given the basepoint 0.

$$\mathbf{1}: X \to M_n(X), \qquad \mathbf{1}(x)(s) = (s, x),$$

$$\mu: M_n(X) \wedge M_n(Y) \to M_n(X \wedge Y),$$

$$\mu(f, g): [n] \xrightarrow{g} [n] \wedge Y \xrightarrow{f \wedge \mathrm{id}} [n] \wedge X \wedge Y, \quad f \in M_n(X),\ g \in M_n(Y).$$

Lemma 2.3. *Given FSP's L_1 and L_2 the composite L_1L_2 has structure maps*

$$\mathbf{1}: X \xrightarrow{\mathbf{1}^{L_2}} L_2(X) \xrightarrow{\mathbf{1}^{L_1}} L_1L_2(X),$$

$$\mu: L_1L_2(X) \wedge L_1L_2(Y) \xrightarrow{\mu^{L_1}} L_1(L_2(X) \wedge L_2(Y)) \xrightarrow{L_1\mu^{L_2}} L_1L_2(X \wedge Y).$$

Thus given any FSP L we may form new FSPs such as M_nL and $L[G] = \widetilde{G}L$.

Let $\mathscr{I}$ be the category with objects the finite sets $\mathbf{n} = \{1, \ldots, n\}$ and morphisms the injective maps, and consider the functors $\mathrm{G}_i[L,m] : \mathscr{I}^{i+1} \to \mathbf{Simp}_*$ with

$$\mathrm{G}_i[L,m](\mathbf{n}_0, \ldots, \mathbf{n}_i) = \mathrm{Map}_*(S^{n_0} \wedge \cdots \wedge S^{n_i}, L(S^{n_0}) \wedge \cdots \wedge L(S^{n_i}) \wedge S^m).$$

Here $\mathrm{Map}_*(-,-)$ is the simplicial mapping space, given on simplicial sets X and Y by $\sin_\bullet \mathrm{Map}_*(|X|,|Y|)$. We let

$$\mathrm{THH}_i(L,m) = \operatorname*{hocolim}_{\mathscr{I}^{i+1}} \mathrm{G}_i[L,m],$$

and make this the i-simplices in a cyclic space $\mathrm{THH}(L,m)$ with face and degeneracy maps induced by the multiplication and unit in L as in (1.1).

Definition 2.4. The topological Hochschild spectrum $\mathrm{TH}(L)$ is the simplicial spectrum given in degree m as $\mathrm{THH}(L,m)$ and with spectrum maps induced by the obvious maps $\mathrm{G}_i[L,m] \wedge S^1 \to \mathrm{G}_i[L,m+1]$.

We shall often need the following approximation lemma due to Bökstedt.

Lemma 2.5 (Bökstedt [5, 1.5] and Madsen [13, 2.3.7]). *Let L be a connected FSP. Given $k \geq 0$ there exists $n \geq 0$ such that*

$$\mathrm{G}_i[L,m](\mathbf{n}, \ldots, \mathbf{n}) \to \mathrm{THH}_i(L,m)$$

is a k-equivalence.

Remark 2.6. For the purpose of this paper it suffices to use a naive notion of a spectrum. Thus for us a spectrum E is simply a sequence of pointed simplicial sets E_m together with simplicial maps $E_m \wedge S^1 \to E_{m+1}$. A map of spectra $f : E \to F$ is a sequence of maps $f_m : E_m \to F_m$ which commutes with the structure maps. The homotopy groups are defined as $\pi_i(E) = \varinjlim \pi_{i+m}(E_m)$ and f is called a stable weak equivalence if the induced maps $f_* : \pi_i(E) \to \pi_i(F)$ are isomorphisms.

To construct the new model $\mathrm{TH}^{\dagger}(L)$ we first consider the following modification of the functor $\mathrm{G}_i[L,m]$:

$$\mathrm{G}_i^+[L,m](\mathbf{n}_0, \ldots, \mathbf{n}_i) = \mathrm{Map}_*(S^{n_0} \wedge \cdots \wedge S^{n_i}, \Gamma_i^+(L(S^{n_0}) \wedge \cdots \wedge L(S^{n_i}) \wedge S^m)).$$

We then define $\mathrm{THH}_i^+(L,m) = \operatorname{hocolim}_{\mathscr{I}^{i+1}} \mathrm{G}_i^+[L,m]$ and make this into a simplicial space as before, but now taking the simplicial structure of Γ^+ into account. The approximation Lemma 2.5 still holds.

Lemma 2.7. *Let L be a connected FSP. Given $k \geq 0$ there exists $n \geq 0$ such that*

$$\mathrm{G}_i^+[L,m](\mathbf{n}, \ldots, \mathbf{n}) \to \mathrm{THH}_i^+(L,m)$$

is a k-equivalence.

Definition 2.8. We let $\mathrm{TH}^+(L)$ be the simplicial spectrum given in degree m as $\mathrm{THH}^+(L,m)$ and with spectrum maps induced by the obvious maps

$$\mathrm{G}_i^+[L,m]\wedge S^1 \to \mathrm{G}_i^+[L,m+1].$$

Note that when i is fixed and X is a simplicial set then $\Gamma_i^+(X)$ is a simplicial set and there is an inclusion

$$X \to \Gamma_i^+(X), \qquad \mathbf{x} \mapsto [(\mathbf{1}_1,\dots,\mathbf{1}_1);\mathbf{x}].$$

Lemma 2.9 (Barratt and Eccles [3, Section 6]). *Assume that X is $(n-1)$-connected for $n\geq 1$. Then the inclusion $X\to\Gamma_i^+X$ is $(2n-1)$-connected.*

Lemma 2.10. *There is a natural equivalence* $\mathrm{TH}(L)\to\mathrm{TH}^+(L)$.

Proof. The map in question is induced by the inclusion

$$L(S^{n_0})\wedge\cdots\wedge L(S^{n_i})\wedge S^m \to \Gamma_i^+(L(S^{n_0})\wedge\cdots\wedge L(S^{n_i})\wedge S^m).$$

It follows from the approximation Lemmas 2.5 and 2.7 and Lemma 2.9 that this gives a homotopy equivalence $\mathrm{THH}_i(L,m)\to\mathrm{THH}_i^+(L,m)$. The result now follows from the realization lemma for bisimplicial sets. □

3. Morita equivalence

In this section we construct the trace map $\mathrm{tr}:\mathrm{TH}(M_nL)\to\mathrm{TH}^+(L)$. To show that it is an equivalence, we introduce an intermediate functor W_n that fits in a commutative diagram

$$\begin{array}{ccc}
\mathrm{TH}(M_n(L)) & \xrightarrow{\mathrm{tr}} & \mathrm{TH}^+(L)\\
\uparrow{\scriptstyle i} & & \uparrow\\
\mathrm{TH}(W_n(L)) & \xrightarrow{\mathrm{tr}} & \mathrm{TH}(L),
\end{array}$$

and we shall prove that the vertical maps and the lower horizontal map are equivalences.

Definition 3.1. Let W_n be the functor on based spaces with $W_n(X)=[n]\wedge X\wedge[n]$ and multiplication

$$\mu: W_n(X)\wedge W_n(Y)\to W_n(X\wedge Y),$$

$$\mu((s_1,x,t_1),(s_2,y,t_2))=\begin{cases}(s_1,x,y,t_2), & t_1=s_2\neq 0,\\ * & \text{otherwise.}\end{cases}$$

We note that μ is associative, and call W_n a pre-FSP. We may form $\mathrm{G}_i[W,m]$ as in the case of a FSP and define

$$\mathrm{THH}_i(W,m) = \operatorname*{hocolim}_{\mathcal{I}^{i+1}} \mathrm{G}_i[W,m].$$

Since W_n has no unit there is no degeneracy maps, but we can still make $[i] \mapsto \mathrm{THH}_i(W,m)$ into a pre-simplicial space, i.e. a simplicial set without degeneracy operators. We thus get a pre-simplicial spectrum $\mathrm{TH}(W_n)$, or more generally $\mathrm{TH}(W_nL)$ for any FSP L.

If we think of $M_n(X)$ as matrices with at most one entry different from zero in each column, then $W_n(X)$ corresponds to matrices with at most one entry different from zero. There is an inclusion $i : W_n(X) \to M_n(X)$.

$$i((s,x,t)) : [n] \to [n] \wedge X, \qquad i((s,x,t))(u) = \begin{cases} (s,x), & t = u \neq 0, \\ * & \text{otherwise.} \end{cases}$$

Lemma 3.2. *Let L be any FSP. Then $i : W_n \to M_n$ induces an equivalence*

$$i : \mathrm{TH}(W_nL) \to \mathrm{TH}(M_nL).$$

Proof. As spaces

$$W_n(X) = \bigvee_{t=1}^{n} \bigvee_{s=1}^{n} X \quad \text{and} \quad M_n(X) = \prod_{t=1}^{n} \bigvee_{s=1}^{n} X,$$

and i is just the inclusion. Since i is $(2m-1)$-connected when X is $(m-1)$-connected the lemma follows from the approximation Lemma 2.5. $\square$

There is a trace map

$$\mathrm{tr} : W_n(X_0) \wedge \cdots \wedge W_n(X_i) \to X_0 \wedge \cdots \wedge X_i,$$

$$\mathrm{tr}((s_0,x_0,t_0),\ldots,(s_i,x_i,t_i)) = \begin{cases} (x_0,\ldots,x_i), & \text{if } t_0 = s_1, \ldots, t_i = s_0, \text{ all} \neq 0, \\ * & \text{otherwise.} \end{cases} \tag{3.1}$$

This map induces a natural transformation $\mathrm{tr} : \mathrm{G}_i[W_nL,m] \to \mathrm{G}_i[L,m]$ and it is easy to see that there is an induced map of spectra

$$\mathrm{tr} : \mathrm{TH}(W_nL) \to \mathrm{TH}(L).$$

As in the linear case there also is a map in the other direction.

Definition 3.3. $\mathrm{inc} : I \to W_n$ is the map of pre-FSP's given by

$$\mathrm{inc} : X \to W_n(X), \qquad \mathrm{inc}(x) = (1,x,1).$$

We again get a map of spectra $\mathrm{inc} : \mathrm{TH}(L) \to \mathrm{TH}(W_nL)$ and we have the following.

Lemma 3.4. $\mathrm{tr}:\mathrm{TH}(W_nL)\to\mathrm{TH}(L)$ *is an equivalence of spectra with homotopy inverse the inclusion (Definition 3.3).*

Proof. It is obvious that $\mathrm{tr}\circ\mathrm{inc}$ is the identity on $\mathrm{THH}(L,m)$. Thus to show that tr is a homotopy equivalence, it suffices to show that $\|\mathrm{inc}\circ\mathrm{tr}\|$ is a homotopy equivalence on $\|\mathrm{THH}(W_nL,m)\|$. (We use $\|-\|$ to mean the realization of a pre-simplicial space, cf. [17, Appendix].) Since $\mathrm{THH}(W_nL,m)$ is simply connected when $m\geq 2$ it is enough to show that $\|\mathrm{inc}\circ\mathrm{tr}\|$ induces an isomorphism on homology. For this purpose we can adapt the pre-simplicial homotopy from the linear case [12, 1.2.4] to the topological setting. Define natural transformations

$$\begin{aligned} h_v &: W_nL(X_0)\wedge\cdots\wedge W_nL(X_i)\\ &\to W_nL(X_0)\wedge\cdots\wedge W_nL(X_v)\wedge W_nL(S^0)\wedge\cdots\wedge W_nL(X_i),\quad 0\leq v\leq i,\end{aligned}$$

$$h_v((s_0,x_0,t_0),\dots,(s_i,x_i,t_i))$$
$$=\begin{cases}((s_0,x_0,1),\dots,(1,x_v,1),(1,\mathbf{1}^L(u),t_v),(s_{v+1},x_{v+1},t_{v+1}),\dots,(s_i,x_i,t_i)),\\ \qquad \text{if } t_0=s_1,\dots,t_{v-1}=s_v,\ \text{all}\neq 0,\\ * \quad \text{otherwise.}\end{cases}$$

Here $u\in S^0$ is the element different from the basepoint.

As in the construction of the degeneracy maps in THH we get induced maps $h_v:\mathrm{THH}_i(W_nL,m)\to\mathrm{THH}_{i+1}(W_nL,m)$ by using the functorial properties of hocolim. It is now elementary though tedious to check that this is a pre-simplicial homotopy (in the sense of [12, 1.0.8]) from id to $\mathrm{inc}\circ\mathrm{tr}$.

To finish the proof, we recall that the homology of $\|\mathrm{THH}(W_nL,m)\|$ can be calculated using the chain complex $(\mathbb{Z}_*\mathrm{THH}(W_nL,m),\sum(-1)^v d_{v*})$ associated with the pre-simplicial space $\mathrm{THH}(W_nL,m)$. Then $h=\sum(-1)^v h_{v*}$ is a chain homotopy from the identity to $\mathrm{inc}_*\circ\mathrm{tr}_*$. □

We next construct a natural transformation

$$\mathrm{tr}:M_n(X_0)\wedge\cdots\wedge M_n(X_i)\to\Gamma_i^+(X_0\wedge\cdots\wedge X_i),\tag{3.2}$$

analogous to the trace map of cyclic nerves discussed in Section 1. Substituting $L(X_v)$ for X_v in (3.2), this natural transformation will then induce the trace map $\mathrm{tr}:\mathrm{TH}(M_nL)\to\mathrm{TH}^+(L)$.

First note that as $[n]=\mathbf{n}\cup\{*\}$, there is an inclusion

$$M_n(X)=\mathrm{Map}_*([n],[n]\wedge X)\to\mathrm{Map}(\mathbf{n}\times\mathbf{n},X),\qquad f\mapsto(f_{ij}).$$

If we interpret basepoints as zero elements the inclusion of the product $\mu(f,g)\in M_n(X\wedge Y)$ can be written as

$$\mu(f,g)_{ij}=\sum_s(f_{is},g_{sj})\in X\wedge Y.$$

We proceed as in (1.6). Given $(f^0,\dots,f^i)\in M_n(X_0)\wedge\cdots\wedge M_n(X_i)$, we let

$$S=\{(j_0,\dots,j_i)\in \mathbf{n}^{i+1}: f^0_{j_i j_0}\neq *,\dots,f^i_{j_{i-1}j_i}\neq *\}.$$

Choose some arbitrary ordering of S, $\rho=(\rho_0,\dots,\rho_i):\mathbf{m}\to S$, and for $v=0,\dots,i$ let $\alpha_v\in\Sigma_m$ be determined by

$$\rho_v(\alpha_v^{-1}(1))<\cdots<\rho_v(\alpha_v^{-1}(m)).$$

We then define

$$\operatorname{tr}(f^0,\dots,f^i)=[\boldsymbol{\alpha};\mathbf{f}(1),\dots,\mathbf{f}(m)]\in\Gamma_i^+(X_0\wedge\cdots\wedge X_i), \tag{3.3}$$

where $\boldsymbol{\alpha}=(\alpha_0,\dots,\alpha_i)$, and $\mathbf{f}(v)=(f^0_{\rho_i(v)\rho_0(v)},\dots,f^i_{\rho_{i-1}(v)\rho_i(v)})$.

We leave it for the reader to check that this is independent of the ordering of S, and that tr is simplicial in $X_0,\dots,X_i$.

There are natural "face" maps

$$d_v: M_n(X_0)\wedge\cdots\wedge M_n(X_i)\to M_n(X_0)\wedge\cdots\wedge M_n(X_v\wedge X_{v+1})\wedge\cdots\wedge M_n(X_i),$$

$$d_v^+:\Gamma_i^+(X_0\wedge\cdots\wedge X_i)\to\Gamma_{i-1}^+(X_0\wedge\cdots\wedge(X_v\wedge X_{v+1})\wedge\cdots\wedge X_i)$$

for $v=0,\dots,i-1$, and

$$d_i: M_n(X_0)\wedge\cdots\wedge M_n(X_i)\to M_n(X_i\wedge X_0)\wedge\cdots\wedge M_n(X_{i-1}),$$
$$d_i^+:\Gamma_i^+(X_0\wedge\cdots\wedge X_i)\to\Gamma_{i-1}^+(X_i\wedge X_0\wedge\cdots\wedge X_{i-1}).$$

Similarly, we have "degeneracies",

$$s_v: M_n(X_0)\wedge\cdots\wedge M_n(X_i)\to M_n(X_0)\wedge\cdots\wedge M_n(X_v)\wedge M_n(S^0)\wedge\cdots\wedge M_n(X_i),$$
$$s_v^+:\Gamma_i^+(X_0\wedge\cdots\wedge X_i)\to\Gamma_{i+1}^+(X_0\wedge\cdots\wedge X_v\wedge S^0\wedge\cdots\wedge X_i)$$

and "cyclic" operators

$$t_i: M_n(X_0)\wedge\cdots\wedge M_n(X_i)\to M_n(X_i)\wedge M_n(X_0)\wedge\cdots\wedge M_n(X_{i-1}),$$
$$t_i^+:\Gamma_i^+(X_0\wedge\cdots\wedge X_i)\to\Gamma_i^+(X_i\wedge X_0\wedge\cdots\wedge X_{i-1}).$$

Lemma 3.5. *The trace map* (3.2) *satisfies* $d_v^+\circ\operatorname{tr}=\operatorname{tr}\circ d_v$, $s_v^+\circ\operatorname{tr}=\operatorname{tr}\circ s_v$ *and* $t_i^+\circ\operatorname{tr}=\operatorname{tr}\circ t_i$ *for* $v=0,\dots,i$.

Theorem 3.6. *The natural transformation* tr *from* (3.2) *induces a cyclic map*

$$\operatorname{tr}:\mathrm{THH}(M_nL,m)\to\mathrm{THH}^+(L,m).$$

This is a homotopy equivalence, giving a degree-wise equivalence of spectra

$$\operatorname{tr}:\mathrm{TH}(M_nL)\to\mathrm{TH}^+(L).$$

Proof. From Lemma 3.5, it follows easily that tr is a cyclic map. To see that it is a homotopy equivalence note that we have a commutative diagram of pre-simplicial spaces

$$\begin{tikzcd} \mathrm{THH}(M_n(L), m) \arrow[r, "\mathrm{tr}"] & \mathrm{THH}^+(L, m) \\ \mathrm{THH}(W_n(L), m) \arrow[u, "i"] \arrow[r, "\mathrm{tr}"] & \mathrm{THH}(L, m). \arrow[u] \end{tikzcd}$$

From Lemmas 3.2, 3.4 and 2.10 we know that the other three arrows induces a homotopy equivalence after pre-simplicial realization. Therefore, the same holds for the trace map. By [17, Appendix] the quotient map

$$\|X\| \to |X|$$

is a homotopy equivalence when X is a good simplicial space. In our case the simplicial spaces comes as realizations of pre-simplicial sets, and since all simplicial spaces arising in this way are good, the result follows. □

Remark 3.7. We can use the monadic structure of Γ^+

$$\mu : \Gamma^+\Gamma^+(X) \to \Gamma^+(X), \quad [3, 3.5]$$

to obtain a trace map equivalence $\mathrm{THH}^+(M_nL) \to \mathrm{THH}^+(L)$. This is nice from a formal point of view, but of no importance for the calculations we are after.

4. The restriction map in TH$^+$

Let G be a discrete group, $K \subseteq G$ a subgroup with finite index in G and choose a set of representatives for the left cosets

$$G/K = \{\gamma_1 K, \ldots, \gamma_n K\}. \tag{4.1}$$

There is a left action of G on G/K and a group element $\sigma \in G$ gives rise to two functions

$$j(\sigma) : [n] \to [n], \quad \text{and} \quad \tilde{\sigma} : [n] \to K_+ \tag{4.2}$$

by the requirement that $\sigma\gamma_s = \gamma_{j(\sigma)(s)}\tilde{\sigma}(s)$.

Definition 4.1. $i^\sharp : \widetilde{G} \to M_n\widetilde{K}$ is the map of FSP's defined by

$$i^\sharp_X(x, \sigma) : [n] \to [n] \wedge X \wedge K_+, \quad s \mapsto (j(\sigma)(s), x, \tilde{\sigma}(s)).$$

By composing with Morita equivalence we get the restriction map

$$\mathrm{Res} : \mathrm{TH}(L[G]) \to \mathrm{TH}(M_n(L[K])) \to \mathrm{TH}^+(L[K]). \tag{4.3}$$

I next have to discuss smash products of spectra. Since we are working in a "naive" category of (pre)spectra, we shall also use an ad hoc construction of the smash product, cf. [2, Section 4]. First define two functions $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ as follows:

$$\alpha(n) = |\{x \in \mathbb{N} : x \notin 2\mathbb{N};\ x < n\}|,$$

$$\beta(n) = |\{x \in \mathbb{N} : x \in 2\mathbb{N};\ x < n\}|.$$

Notice that $\alpha(n) + \beta(n) = n$ for all n. For spectra E and F we then define the smash product $E \wedge F$ as having $(E \wedge F)_n = E_{\alpha(n)} \wedge F_{\beta(n)}$, and structure maps

$$E_{\alpha(n)} \wedge F_{\beta(n)} \wedge S^1 \to E_{\alpha(n)} \wedge F_{\beta(n)+1} = E_{\alpha(n+1)} \wedge F_{\beta(n+1)} \qquad \text{for } n \in 2\mathbb{N}$$

$$E_{\alpha(n)} \wedge F_{\beta(n)} \wedge S^1 \to E_{\alpha(n)} \wedge S^1 \wedge F_{\beta(n)} \to E_{\alpha(n)+1} \wedge F_{\beta(n)} = E_{\alpha(n+1)} \wedge F_{\beta(n+1)}$$

for $n \notin 2\mathbb{N}$.

For a simplicial set X let $\Sigma^\infty(X)$ denote the suspension spectrum of X, that is $\Sigma^\infty(X)_n = X \wedge S^n$. Also define $\Gamma^+\Sigma^\infty(X)$ to be the spectrum with $\Gamma^+\Sigma^\infty(X)_n = \Gamma^+(X \wedge S^n)$ and with the obvious structure maps. Notice that by Lemma 2.9 there is a stable equivalence $\Sigma^\infty(X) \to \Gamma^+\Sigma^\infty(X)$.

Lemma 4.2. *There are stable equivalences of spectra*

$$\mathrm{TH}(L) \wedge \Sigma^\infty(\mathrm{N}^{\mathrm{cy}}_\wedge(G_+)) \simeq \mathrm{TH}(L[G]),$$

$$\mathrm{TH}(L) \wedge \Gamma^+\Sigma^\infty(\mathrm{N}^{\mathrm{cy}}_\wedge(K_+)) \simeq \mathrm{TH}^+(L[K]).$$

Proof. We concentrate on the second equivalence since the proof of the first is similar. In degree $\alpha(n) + \beta(n)$ the equivalence is induced by the composite map

$$\begin{aligned}
&\mathrm{Map}_*(S^{n_0} \wedge \cdots \wedge S^{n_i}, L(S^{n_0}) \wedge \cdots \wedge L(S^{n_i}) \wedge S^{\alpha(n)}) \wedge \Gamma_i^+(\mathrm{N}^{\mathrm{cy}}_{\wedge\, i}(K_+) \wedge S^{\beta(n)}) \\
&\to \mathrm{Map}_*(S^{n_0} \wedge \cdots \wedge S^{n_i}, L(S^{n_0}) \wedge \cdots \wedge L(S^{n_i}) \wedge S^{\alpha(n)} \wedge \Gamma_i^+(\mathrm{N}^{\mathrm{cy}}_{\wedge\, i}(K_+) \wedge S^{\beta(n)})) \\
&\to \mathrm{Map}_*(S^{n_0} \wedge \cdots \wedge S^{n_i}, \Gamma_i^+(L(S^{n_0}) \wedge \cdots \wedge L(S^{n_i}) \wedge \mathrm{N}^{\mathrm{cy}}_{\wedge\, i}(K_+) \wedge S^{\alpha(n)+\beta(n)})).
\end{aligned}$$

Here we permute the coordinates in $S^{\alpha(n)+\beta(n)}$ so as to get a map of spectra.

Now the first map is $(2\alpha(n)+\beta(n)-1)$-connected and the second is $(\alpha(n)+2\beta(n)-1)$-connected, and so the composite map is approximately $(\alpha(n) + 2\beta(n) - 1)$-connected. By the approximation Lemma 2.7 this also holds for the induced map

$$\mathrm{THH}_i(L, \alpha(n)) \wedge \Gamma_i^+(\mathrm{N}^{\mathrm{cy}}_{\wedge\, i}(K_+) \wedge S^{\beta(n)}) \to \mathrm{THH}_i^+(L[K], \alpha(n) + \beta(n)).$$

By the realization lemma for simplicial spaces [18, 2.1.1] we thus get a map of spectra

$$\mathrm{TH}(L) \wedge \Gamma^+\Sigma^\infty(\mathrm{N}^{\mathrm{cy}}_\wedge(K_+)) \to \mathrm{TH}^+(L[K])$$

which in degree $\alpha(n) + \beta(n)$ is approximately $(\alpha(n) + 2\beta(n) - 1)$-connected. It is therefore a stable equivalence. $\square$

Now consider G_+ as a pointed monoid in the sense of Section 1. Using the trace map (1.6), we get a map of pointed monoids, similar to the restriction map (4.3):

$$\widetilde{\mathrm{Res}}:\mathrm{N}^{\mathrm{cy}}_{\wedge}(G_+)\xrightarrow{\tilde{i}}\mathrm{N}^{\mathrm{cy}}_{\wedge}(M_n(K_+))\xrightarrow{\mathrm{tr}}\Gamma^+(\mathrm{N}^{\mathrm{cy}}_{\wedge}(K_+)).$$

For future reference we give an explicit formula for this map. Given $\boldsymbol{\sigma}=(\sigma_0,\dots,\sigma_i)$ in G^{i+1} (simplicial degree i) we introduce the notation

$$\boldsymbol{\sigma}[v,i]=\begin{cases}\sigma_v\cdot\ldots\cdot\sigma_i & \text{for } 0\le v\le i\\ \mathbf{1} & \text{for } v>i\end{cases}\in G. \tag{4.4}$$

Let $S=\{s\in\mathbf{n}:j(\boldsymbol{\sigma}[0,i])(s)=s\}$ and define $\rho\in\mathcal{M}(\mathbf{m},\mathbf{n})$ by the condition that $\rho(\mathbf{m})=S\subseteq\mathbf{n}$. We have the restriction map ρ^* from 1.6 and we let $\alpha_v=\rho^*(j(\boldsymbol{\sigma}[v+1,i]))\in\Sigma_m$ for $v=0,\dots,i$. Then

$$\widetilde{\mathrm{Res}}(\boldsymbol{\sigma})=[(\alpha_0,\dots,\alpha_i);(\mathbf{y}_1,\dots,\mathbf{y}_m)]\in\Gamma_i^+(\mathrm{N}^{\mathrm{cy}}_{\wedge,i}(K_+)), \tag{4.5}$$

where

$$\mathbf{y}_s=(\tilde{\sigma}_0(j(\boldsymbol{\sigma}[1,i])(\rho(s))),\tilde{\sigma}_1(j(\boldsymbol{\sigma}[2,i])(\rho(s))),\dots,\tilde{\sigma}_i(\rho(s))).$$

The stabilization of $\widetilde{\mathrm{Res}}$ is a map of spectra $\Sigma^\infty(\mathrm{N}^{\mathrm{cy}}_{\wedge}(G_+))\to\Gamma^+\Sigma^\infty(\mathrm{N}^{\mathrm{cy}}_{\wedge}(K_+))$, which we also denote by $\widetilde{\mathrm{Res}}$. Explicitly this is given in degree n as

$$\widetilde{\mathrm{Res}}:\mathrm{N}^{\mathrm{cy}}_{\wedge}(G_+)\wedge S^n\to\Gamma^+(\mathrm{N}^{\mathrm{cy}}_{\wedge}(K_+))\wedge S^n\to\Gamma^+(\mathrm{N}^{\mathrm{cy}}_{\wedge}(K_+)\wedge S^n),$$

and it is easy to check the following.

Proposition 4.3. *There is a commutative diagram of spectra, where the vertical maps are the equivalences from Lemma* 4.2.

$$\begin{CD}
\mathrm{TH}(L[G]) @>{\mathrm{Res}}>> \mathrm{TH}^+(L[K])\\
@A{\simeq}AA @A{\simeq}AA\\
\mathrm{TH}(L)\wedge\Sigma^\infty(\mathrm{N}^{\mathrm{cy}}_{\wedge}(G_+)) @>{\mathrm{id}\wedge\widetilde{\mathrm{Res}}}>> \mathrm{TH}(L)\wedge\Gamma^+\Sigma^\infty(\mathrm{N}^{\mathrm{cy}}_{\wedge}(K_+))
\end{CD}$$

5. Simplicial transfers

To each n-sheeted covering $p:E\to B$ of topological spaces there is a stable transfer $\mathrm{trf}:\Sigma^\infty(B_+)\to\Sigma^\infty(E_+)$, see [1]. Indeed, consider the associated Σ_n-principal bundle

$$\mathrm{P}(E)=\{(x_1,\dots,x_n)\in E^n:p(x_s)=p(x_t),\,x_s\ne x_t\text{ for }s\ne t\}.$$

There is a Σ_n equivariant map into the universal Σ_n-bundle $\mathrm{P}(E)\to\mathrm{E}\Sigma_n$, uniquely determined up to equivariant homotopy. The inclusion $\mathrm{P}(E)\to E^n$ is also Σ_n equivariant, so we have the equivariant map $\mathrm{P}(E) \to \mathrm{E}\Sigma_n \times E^n$. Since $\mathrm{P}(E)/\Sigma_n \equiv B$ we may consider the composite

$$B \equiv \mathrm{P}(E)/\Sigma_n \to \mathrm{E}\Sigma_n \times_{\Sigma_n} E^n \to \mathrm{E}\Sigma_n \times_{\Sigma_n} Q(E_+)^n \xrightarrow{\Theta} Q(E_+).$$

Here we have taken as a model for $\mathrm{E}\Sigma_n$ the space of n little cubes $\mathscr{C}_\infty(n)$, and Θ is the operad action of $\mathscr{C}_\infty$ on $Q(E_+) = \varinjlim \Omega^n \Sigma^n(E_+)$ (for the definition of operads see [14]). The transfer is then the adjoint of the above map.

We shall need a simplicial analogue of this. Given a map of simplicial sets $p : X \to A$ and an element $a \in A_m$ we may form the pullback:

$$\begin{tikzcd} \bar{a}^* X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[m] \arrow[r, "\bar{a}"] & A, \end{tikzcd}$$

where $\bar{a} : \Delta[m] \to A$ is the characteristic simplicial map with $\bar{a}(\mathbf{1}_m) = a$.

Definition 5.1 (*Lamotke* [10]). Let Z be a discrete set. The map p is called a simplicial covering with fiber Z if for every $a \in A$ there is a simplicial isomorphism $\hat{a}$ such that the diagram

$$\begin{tikzcd} \Delta[m] \times Z \arrow[r, "\hat{a}"] \arrow[d] & \bar{a}^* X \arrow[d] \\ \Delta[m] \arrow[r, equal] & \Delta[m] \end{tikzcd}$$

commutes. If $|Z| = n$ then p is called an n-sheeted covering.

For example, if a discrete group G acts freely on a simplicial set X and $K \subseteq G$ is a subgroup with $|G/K| = n$, then the quotient map $X/K \to X/G$ is an n-sheeted covering. A principal G bundle is a covering of the form $X \to X/G$.

To an n-sheeted covering $p : X \to A$ there is an associated principal Σ_n bundle $\mathrm{P}(X) \to A$, constructed degree-wise:

$$\mathrm{P}(X)_m = \{(x_1, \ldots, x_n) \in X_m^n : p(x_s) = p(x_t), \text{and } x_s \neq x_t \text{ for } s \neq t\}.$$

It is easy to check that $\mathrm{P}(X) \subseteq X^n$ is a simplicial subset, and that Σ_n acts freely on $\mathrm{P}(X)$ with quotient $\mathrm{P}(X)/\Sigma_n \cong A$.

Recall from 1.8 the functor E from sets to simplicial sets. For Π a discrete group this induces a functor from Π-sets to simplicial Π-sets by introducing the diagonal action on $E_iX = X^{i+1}$. Let D_0 be the functor from simplicial Π-sets to Π-sets, which projects on simplicial degree zero.

Lemma 5.2. *E is right adjoint to* D_0 *in the sense that there is a natural bijection of hom sets*

$$\Pi\text{-}\mathbf{Sets}(X_0, Y) \cong \Pi\text{-}\mathbf{Simp}(X, \mathrm{E}Y)$$

for every simplicial Π*-set* X *and every* Π*-set* Y.

Proof. Given a Π map $f_0 : X_0 \to Y$ we get the unique extension to a simplicial Π map $f_i : X_i \to \mathrm{E}_iY$ by letting

$$f_i(x) = (f_0\bar{x}(0), \dots, f_0\bar{x}(i)), \tag{5.1}$$

where $x \in X_i$, and $\bar{x} : \Delta[i] \to X$. $\square$

One advantage of the simplicial approach to the transfer is that the classifying map into $\mathrm{E}\Sigma_n$ becomes very explicit. Given a Σ_n principal bundle P, a choice of representatives $\{v_i\}$ for the Σ_n orbits of P_0 gives a Σ_n equivariant map

$$\lambda_0 : P_0 = \coprod v_i\Sigma_n \to \Sigma_n, \qquad \lambda_0(v_i\sigma) = \sigma,$$

which by Lemma 5.2 then has a unique extension to a Σ_n equivariant simplicial map $\lambda : P \to \mathrm{E}\Sigma_n$. Furthermore, two different choices of representatives give homotopic maps. Indeed, we obtain an equivariant simplicial homotopy $P \times \Delta[1] \to \mathrm{E}\Sigma_n$ by extending the map already given in degree zero: $P_0 \times \{(0), (1)\} \to \Sigma_n$.

Returning to the principal bundle $\mathrm{P}(X) \to A$ we thus get an equivariant map $\mathrm{P}(X) \to \mathrm{E}\Sigma_n$, which is well-defined up to equivariant simplicial homotopy. Of course the inclusion $\mathrm{P}(X) \to X^n$ is also Σ_n equivariant, and so we may form the composite

$$A \cong \mathrm{P}(X)/\Sigma_n \to \mathrm{E}\Sigma_n \times_{\Sigma_n} X^n \to \mathrm{E}\Sigma_n \times_{\Sigma_n} (X_+)^n \to \Gamma^+(X_+). \tag{5.2}$$

The last map is simply induced from the inclusion

$$\mathrm{E}\Sigma_n \times (X_+)^n \to \coprod_{m \geq 0} \mathrm{E}\Sigma_m \times (X_+)^m.$$

Definition 5.3. The stable transfer

$$\mathrm{trf} : \Sigma^\infty(A_+) \to \Gamma^+\Sigma^\infty(X_+).$$

is the stabilization of (5.2). Explicitly, we have in degree m:

$$\mathrm{trf} : A_+ \wedge S^m \to \Gamma^+(X_+) \wedge S^m \to \Gamma^+(X_+ \wedge S^m).$$

It follows from the above discussion that the transfer (Definition 5.3) only depends on an ordering of each fiber of the map in simplicial degree zero $X_0 \to A_0$, and transfers corresponding to different orderings are related by a simplicial homotopy.

Remark 5.4. Since $\Gamma^+(Z)$ is only a model for $Q(Z)$ when Z is connected, we should really map to the group completion $\Gamma(Z)$, [3, Section 4]. However, since $\Gamma^+(Z) \to \Gamma(Z)$ is a homotopy equivalence when Z is connected the corresponding maps of spectra becomes equivalent. As in Remark 3.7 we could use the monadic structure of Γ^+ to get a transfer $\Gamma^+\Sigma^\infty(A_+) \to \Gamma^+\Sigma^\infty(X_+)$, but again this is not important for our purposes.

We shall later need to know how the transfer behaves with respect to disjoint unions of coverings. First assume that we have an n-sheeted covering

$$p = p_1 \coprod p_2 : X_1 \coprod X_2 \to A_1 \coprod A_2$$

that comes as the disjoint union of n-sheeted coverings p_1 and p_2. Then the transfer applies to give a commutative diagram of spectra:

$$\begin{tikzcd}
\Sigma^\infty(A_{1+}) \vee \Sigma^\infty(A_{2+}) \arrow[r, "\mathrm{trf} \vee \mathrm{trf}"] \arrow[d, "\simeq"] & \Gamma^+\Sigma^\infty(X_{1+}) \vee \Gamma^+\Sigma^\infty(X_{2+}) \arrow[d, "\simeq"] \\
\Sigma^\infty(A_+) \arrow[r, "\mathrm{trf}"] & \Gamma^+\Sigma^\infty(X_+).
\end{tikzcd} \tag{5.3}$$

(Of course, we have to make coherent choices.)

Next assume that $p_i : X_i \to A$ are coverings for $i = 1, 2$, and let

$$p = \{p_1, p_2\} : X_1 \coprod X_2 \to A$$

be the corresponding $(n_1 + n_2)$-sheeted covering. The transfer applies to give a commutative diagram of spectra:

$$\begin{tikzcd}
\Sigma^\infty(A_+) \arrow[r, "\mathrm{trf}"] \arrow[d, equal] & \Gamma^+\Sigma^\infty(X_+) \arrow[d, "\simeq"] \\
\Sigma^\infty(A_+) \arrow[r, "\mathrm{trf} \times \mathrm{trf}"] & \Gamma^+\Sigma^\infty(X_{1+}) \times \Gamma^+\Sigma^\infty(X_{2+}).
\end{tikzcd} \tag{5.4}$$

Finally, let

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "q"] & X \arrow[d, "p"] \\
B \arrow[r, "g"] & A
\end{tikzcd}$$

be a pullback diagram of n-sheeted coverings. There results a commutative diagram of spectra:

$$\begin{tikzcd}
\Sigma^\infty(B_+) \arrow[r, "\mathrm{trf}"] \arrow[d, "\Sigma^\infty g"] & \Gamma^+\Sigma^\infty(Y_+) \arrow[d, "\Gamma^+\Sigma^\infty f"] \\
\Sigma^\infty(A_+) \arrow[r, "\mathrm{trf}"] & \Gamma^+\Sigma^\infty(X_+).
\end{tikzcd} \tag{5.5}$$

Lemma 5.4. *Let $p:X \to A$ be a simplicial n-sheeted covering. Then the realization $|p|:|X| \to |A|$ is a topological covering, and the usual transfer $\Sigma^\infty(|X|_+) \to \Sigma^\infty(|A|_+)$ is equivalent to the realization of the above simplicial transfer.*

Proof. For the fact that $|P|:|X| \to |A|$ is a covering see e.g. [10, Ch. 3]. To compare the two definition of the transfer we shall use the theory of operads as developed in [14]. First, Γ^+ can be interpreted as the monad corresponding to the operad consisting of the spaces $|\mathrm{E}\Sigma_n|$, see [14, 15.1]. There is an action of $|\mathrm{E}\Sigma_n|$ on $|\Gamma^+(X_+)| = \Gamma^+(|X_+|)$ and it is easy to see that the composite

$$|A| \cong \mathrm{P}(|X|)/\Sigma_n \to |\mathrm{E}\Sigma_n| \times_{\Sigma_n} \Gamma^+(|X|_+)^n \to \Gamma^+(|X|_+)$$

is precisely the realization of (5.2). Let C_∞ be the monad corresponding to the little cubes operad $\mathscr{C}_\infty$. Then the usual transfer is the adjoint to the map

$$\begin{aligned}|A| \cong \mathrm{P}(|X|)/\Sigma_n &\to \mathscr{C}_\infty(n) \times_{\Sigma_n} |X|^n \\ &\to \mathscr{C}_\infty(n) \times_{\Sigma_n} \mathrm{C}_\infty(|X|_+)^n \\ &\to \mathrm{C}_\infty(|X|_+) \xrightarrow{\alpha} Q(|X|_+).\end{aligned}$$

To compare the two transfers we form the product of these two operads and consider the corresponding monads $\mathrm{C}_\infty \times \Gamma^+$, [14, 3.8]. Then we get a commutative diagram

$$\begin{tikzcd}
|E\Sigma_n| \times_{\Sigma_n} \Gamma^+(|X|_+)^n \arrow[r] & \Gamma^+(|X|_+) \\
(\mathscr{C}_\infty(n) \times |\mathrm{E}\Sigma_n|) \times_{\Sigma_n} (\mathrm{C}_\infty \times \Gamma^+)(|X|_+)^n \arrow[r] \arrow[u] \arrow[d] & (\mathrm{C}_\infty \times \Gamma^+)(|X|_+) \arrow[u] \arrow[d] \\
\mathscr{C}_\infty(n) \times_{\Sigma_n} \mathrm{C}_\infty(|X|_+)^n \arrow[r] & \mathrm{C}_\infty(|X|_+).
\end{tikzcd}$$

By [14, Proposition A.2] the vertical maps are weak homotopy equivalences and the lemma follows. □

The next example relates our simplicial transfer with the usual transfer in singular homology. It also illustrates a recurrent theme in this paper: *in a non-commutative context the right combinatorial substitute for summation is linear ordering.*

Example 5.5. Let $p:X \to A$ be an n-sheeted covering of topological spaces, and consider the induced n-sheeted simplicial covering $\sin X \to \sin A$. The construction of the transfer requires an ordering of each fiber in the map of sets $\sin_0 X \to \sin_0 A$, and the outcome is a map

$$\mathrm{trf}: \Gamma^+(\sin A_+) \to \Gamma^+(\sin X_+).$$

(Here we use the monadic structure of Γ^+.) On the other hand, we have the usual transfer in singular homology. This is represented by a simplicial map

$$\mathrm{trf}: \mathbb{Z}(\sin A) \to \mathbb{Z}(\sin X),$$

obtained by lifting singular chains in A to singular chains in X, cf. [1, Section 5]. These two transfers are related by the commutative diagram

$$\begin{array}{ccc} \Gamma^+(\sin A_+) & \xrightarrow{\mathrm{trf}} & \Gamma^+(\sin X_+) \\ \downarrow & & \downarrow \\ \mathbb{Z}(\sin A) & \xrightarrow{\mathrm{trf}} & \mathbb{Z}(\sin X), \end{array}$$

where the vertical maps are the Hurewicz homomorphisms, induced from the projection $\mathrm{E}\Sigma_n \times \sin X^n \to \sin X^n \to \mathbb{Z}(\sin X)$.

Let us now consider the n-sheeted covering $p: X/K \to X/G$, where G is a discrete group that acts freely on X, and $K \subseteq G$ is a subgroup of index n. The classifying map $\mathrm{P}(X/K) \to \mathrm{E}\Sigma_n$ is constructed after choice of representatives for each Σ_n orbit in $\mathrm{P}_0(X/K)$, or what amounts to the same, choice of a specific ordering of each fiber of the projection $X_0/K \to X_0/G$. This amounts to

(i) Choice of coset representatives for G/K (as in (4.1)).

(ii) Choice of a point $r(xG) \in xG$, i.e. of a map $r: X_0 \to X_0$, constant on G orbits. These data determine an ordering of the fiber over $xG \in X_0/G$, namely

$$p^{-1}(xG) = \{r(x)\gamma_1 K, \dots, r(x)\gamma_n K\}.$$

The choice in (ii) gives a map $q: X_0 \to G$ by letting

$$r(x)q(x) = x, \quad x \in X_0. \tag{5.6}$$

In degree zero $\lambda_0 : \mathrm{P}_0 \to \Sigma_n$ then has $\lambda_0(x\gamma_1 K, \dots, x\gamma_n K) = jq(x)$, for $x \in X_0$, where $jq(x)$ is given by the G action on G/K, cf. 4.2. Now it follows from (5.1) that the transfer

$\mathrm{trf}: X/G \to \Gamma^+(X/K_+)$ is explicitly given by

$$\mathrm{trf}(xG) = [(jq\bar{x}(0), \dots, jq\bar{x}(i));\ (x\gamma_1 K, \dots, x\gamma_n K)], \quad x \in X_i. \tag{5.7}$$

6. Calculation of the restriction map in terms of transfers

As in Section 4 consider an index n subgroup $K \subseteq G$. In this section we prove Theorems A and B from the introduction by comparing the combinatorial descriptions of the restriction and transfer maps given in Sections 4 and 5, respectively. The proof is in two steps. Firstly, we reduce the problem to the study of the transfer corresponding to the covering $\mathrm{E}G \times_K G^{\mathrm{ad}} \to \mathrm{E}G \times_G G^{\mathrm{ad}}$, where G acts on $G^{\mathrm{ad}} = G$ by conjugation. Secondly, we decompose $\mathrm{E}G \times_G G^{\mathrm{ad}}$ into components and get a corresponding decomposition of the transfer.

We let

$$\mathrm{trf}: \mathrm{E}G \times_G G^{\mathrm{ad}}_+ \to \Gamma^+(\mathrm{E}G \times_K G^{\mathrm{ad}}_+)$$

be the (simplicial) transfer of the covering $\mathrm{E}G \times_K G^{\mathrm{ad}} \to \mathrm{E}G \times_G G^{\mathrm{ad}}$. Our choice of coset representatives $G/K = \{\gamma_1 K, \dots, \gamma_n K\}$ determines a K equivariant map

$$f_0 : G \cong \coprod \gamma_i K \to K, \quad f_0(\gamma_i k) = k$$

and by Lemma 5.2 a K-equivariant simplicial map

$$f: \mathrm{E}G \to \mathrm{E}K, \quad f(\sigma_0, \dots, \sigma_i) = (f_0(\sigma_0), \dots, f_0(\sigma_i)).$$

Combining with the projection

$$G \to K_+, \qquad \sigma \mapsto \begin{cases} \sigma & \text{for } \sigma \in K, \\ + & \text{otherwise,} \end{cases}$$

we get a map

$$\kappa : \mathrm{E}G \times_K G^{\mathrm{ad}}_+ \to \mathrm{E}K \times_K K^{\mathrm{ad}}_+,$$

and we want to compare the composite $\Gamma^+(\kappa) \circ \mathrm{trf}$ with the restriction map

$$\widetilde{\mathrm{Res}} = \mathrm{tr} \circ i^\sharp : \mathrm{N}^{\mathrm{cy}}_\wedge(G_+) \to \Gamma^+(\mathrm{N}^{\mathrm{cy}}_\wedge(K)_+)$$

from (4.5). Let $\phi : \mathrm{N}^{\mathrm{cy}}(G) \to \mathrm{E}G \times_G G^{\mathrm{ad}}$ be the simplicial isomorphism given by

$$\phi(\boldsymbol{\sigma}) = (\boldsymbol{\sigma}[1,i], \boldsymbol{\sigma}[2,i], \dots, \boldsymbol{\sigma}[i,i], 1, \boldsymbol{\sigma}[0,i])_G, \tag{6.1}$$

where we use the notation (4.4). Its inverse is

$$\phi^{-1}((\boldsymbol{\sigma}, z)_G) = (\sigma_i z \sigma_0^{-1}, \sigma_0 \sigma_1^{-1}, \dots, \sigma_{i-1} \sigma_i^{-1}).$$

Proposition 6.1. *The diagram*

$$\begin{tikzcd} \mathrm{E}G \times_G G^{\mathrm{ad}}_+ \arrow[d, "\mathrm{trf}"] & \mathrm{N}^{\mathrm{cy}}(G)_+ \arrow[l, "\phi"'] \arrow[d, "i^\sharp"] \\ \Gamma^+(\mathrm{E}G \times_K G^{\mathrm{ad}}_+) \arrow[d, "\Gamma^+(\kappa)"] & \mathrm{N}^{\mathrm{cy}}_\wedge(M_n(K_+)) \arrow[d, "\mathrm{tr}"] \\ \Gamma^+(\mathrm{E}K \times_K K^{\mathrm{ad}}_+) \arrow[r, "\Gamma^+(\phi^{-1})"] & \Gamma^+(\mathrm{N}^{\mathrm{cy}}(K)_+). \end{tikzcd}$$

is commutative.

Proof. We keep the notation from the last paragraph in Section 5, and let

$$r : \mathrm{E}_0 G \times G^{\mathrm{ad}} \to \mathrm{E}_0 G \times G^{\mathrm{ad}}, \qquad r(\sigma_0, z) = (1, \sigma_0 z \sigma_0^{-1}).$$

Then r is constant on G orbits, and the map $q : \mathrm{E}_0 G \times G^{\mathrm{ad}} \to G$ with $q(\sigma_0, z) = \sigma_0$ satisfies $r(\sigma_0, z) q(\sigma_0, z) = (\sigma_0, z)$, cf. (5.6). By (5.7)

$$\mathrm{trf} : \mathrm{E}G \times_G G^{\mathrm{ad}}_+ \to \Gamma^+(\mathrm{E}G \times_K G^{\mathrm{ad}}_+)$$

is given by

$$\mathrm{trf}((\boldsymbol{\sigma}, z)_G) = [(j(\sigma_0), \ldots, j(\sigma_i)), \mathbf{x}_1, \ldots, \mathbf{x}_n],$$

where $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_i)$ and

$$\mathbf{x}_v = (\sigma_0 \gamma_v, \ldots, \sigma_i \gamma_v, \gamma_v^{-1} z \gamma_v)_K \in \mathrm{E}G \times_K G^{\mathrm{ad}}.$$

Clearly, $\gamma_v^{-1} z \gamma_v \in K$ if and only if $j(z)(v) = v$, cf. (4.2), and

$$\kappa(\mathbf{x}_v) = \begin{cases} (\tilde{\sigma}_0(v), \ldots, \tilde{\sigma}_i(v), \tilde{z}(v)) & \text{for } j(z)(v) = v, \\ + & \text{otherwise.} \end{cases}$$

Using the defining relations (4.2), it follows that

$$\Gamma^+ \phi^{-1}(\circ \Gamma^+ \kappa) \circ \mathrm{trf} \circ \phi(\boldsymbol{\sigma}) = [(j(\boldsymbol{\sigma}[1,i]), j(\boldsymbol{\sigma}[2,i]), \ldots, \mathbf{1}_n); (\mathbf{y}_1, \ldots, \mathbf{y}_n)], \tag{6.2}$$

where

$$\mathbf{y}_v = \begin{cases} (\tilde{\sigma}_0(j(\boldsymbol{\sigma}[1,i])(v)), \tilde{\sigma}_1(j(\boldsymbol{\sigma}[2,i])(v)), \ldots, \tilde{\sigma}_i(v)) & \text{for } j(\boldsymbol{\sigma}[0,i])(v) = v, \\ + & \text{otherwise.} \end{cases}$$

We keep $\boldsymbol{\sigma}$ fixed and let $S = \{s \in \mathbf{n} : j(\boldsymbol{\sigma}[0,i])(s) = s\}$ with $|S| = m$. Then define $\rho \in \mathscr{M}(\mathbf{m}, \mathbf{n})$ by the condition that $\rho(\mathbf{m}) = S$, and let $\alpha_v = \rho^*(j(\boldsymbol{\sigma}[v+1]))$ for $v = 0, \ldots, i$.

It follows from (b) in Definition 1.5 that (6.2) is equal to

$$[(\alpha_0,\dots,\alpha_i);(\mathbf{y}_{\rho(1)},\dots,\mathbf{y}_{\rho(m)})], \tag{6.3}$$

which is exactly the formula for $\mathrm{tr}\circ i^{\sharp}$, cf. (4.5). □

It follows from Proposition 4.3 and 6.1 that to calculate the transfer map in THH we just have to determine the map

$$\widetilde{\mathrm{Res}}:\Sigma^\infty(\mathrm{E}G\times_G G^{\mathrm{ad}}_+)\to\Gamma^+\Sigma^\infty(\mathrm{E}G\times_K G^{\mathrm{ad}}_+)\xrightarrow{\Gamma^+\Sigma^\infty\kappa}\Gamma^+\Sigma^\infty(\mathrm{E}K\times_K K^{\mathrm{ad}}_+).$$

Let $\langle G\rangle$ and $\langle K\rangle$ denote the conjugacy classes of K and G, respectively. The decompositions

$$\bigvee_{\omega\in\langle G\rangle}\Sigma^\infty(\mathrm{E}G\times_G\omega_+)\xrightarrow{\simeq}\Sigma^\infty(\mathrm{E}G\times_G G^{\mathrm{ad}}_+)$$
$$\Gamma^+\Sigma^\infty(\mathrm{E}K\times_K K^{\mathrm{ad}}_+)\xrightarrow{\simeq}\prod_{\lambda\in\langle K\rangle}\Gamma^+\Sigma^\infty(\mathrm{E}K\times_K\lambda_+)\quad\text{(weak product)}, \tag{6.4}$$

induce a decomposition of $\widetilde{\mathrm{Res}}$ into maps

$$\widetilde{\mathrm{Res}}^{\lambda}_{\omega}:\Sigma^\infty(\mathrm{E}G\times_G\omega_+)\to\Gamma^+\Sigma^\infty(\mathrm{E}K\times_K\lambda_+). \tag{6.5}$$

We now prove Theorem B from the introduction.

Theorem 6.2. *Let $\omega\in\langle G\rangle$ and $\lambda\in\langle K\rangle$.*
(i) *If $\lambda\nsubseteq\omega$ then $\widetilde{\mathrm{Res}}^{\lambda}_{\omega}\simeq *$.*
(ii) *If $\lambda\subseteq\omega$ then for any $x\in\lambda$ there is a commutative diagram*

$$\begin{tikzcd}
\Sigma^\infty(\mathrm{E}G\times_G\omega_+) \arrow[r, "\widetilde{\mathrm{Res}}^{\lambda}_{\omega}"] \arrow[d, "\simeq"] & \Gamma^+\Sigma^\infty(\mathrm{E}K\times_K\lambda_+) \arrow[d, "\simeq"] \\
\Sigma^\infty(\mathrm{B}C_G(x)_+) \arrow[r, "\mathrm{trf}"] & \Gamma^+\Sigma^\infty(\mathrm{B}C_K(x)_+),
\end{tikzcd}$$

where the vertical maps are equivalences, and the lower horizontal map is the transfer corresponding to the inclusion of centralizers $C_K(x)\to C_G(x)$.

Proof. Let trf_ω be the transfer of the covering $\mathrm{E}G\times_K\omega\to\mathrm{E}G\times_G\omega$. By (5.3) there is a commutative diagram

$$\begin{array}{ccc}
\bigvee_{\omega\in\langle G\rangle} \Sigma^{\infty}(\mathrm{E}G\times_G \omega_+) & \xrightarrow{\simeq} & \Sigma^{\infty}(\mathrm{E}G\times_G G^{\mathrm{ad}}_+) \\
\Big\downarrow \bigvee \mathrm{trf}_{\omega} & & \Big\downarrow \mathrm{trf} \\
\bigvee_{\omega\in\langle G\rangle} \Gamma^{+}\Sigma^{\infty}(\mathrm{E}G\times_K \omega_+) & \xrightarrow{\simeq} & \Gamma^{+}\Sigma^{\infty}(\mathrm{E}G\times_K G^{\mathrm{ad}}_+) \\
\Big\downarrow \bigvee \Gamma^{+}\Sigma^{\infty}\kappa_{\omega} & & \Big\downarrow \Gamma^{+}\Sigma^{\infty}\kappa \\
\bigvee_{\omega\in\langle G\rangle} \Gamma^{+}\Sigma^{\infty}(\mathrm{E}K\times_K K\cap\omega_+) & \xrightarrow{\simeq} & \Gamma^{+}\Sigma^{\infty}(\mathrm{E}K\times_K K^{\mathrm{ad}}_+).
\end{array}$$

Now (i) follows from the definition of κ. To prove (ii) let $\omega\in\langle G\rangle$ be fixed and consider the decomposition

$$\mathrm{E}G\times_K\omega=\coprod \mathrm{E}G\times_K\lambda,$$

where on the right side the union is over all K-conjugacy classes λ in ω. For $\lambda\subseteq\omega$ choose $x\in\lambda$ and consider the diagram

$$\begin{array}{ccc}
\mathrm{E}G\times_K\lambda & \longrightarrow & \mathrm{E}G\times_G\omega \\
\Big\downarrow \simeq & & \Big\downarrow \simeq \\
\mathrm{E}G/C_G(x)\cap K & \longrightarrow & \mathrm{E}G/C_G(x).
\end{array} \tag{6.6}$$

We see that

$$\mathrm{E}G\times_K\lambda\to\mathrm{E}G\times_G\omega \tag{6.7}$$

is a $|C_G(x)/C_K(x)\cap K|$-sheeted covering, and we can apply (5.4) to obtain the commutative diagram

$$\begin{tikzcd}
\Sigma^\infty(\mathrm{E}G \times_G \omega_+) \arrow[r, equal] \arrow[d, "\mathrm{trf}"] & \Sigma^\infty(\mathrm{E}G \times_G \omega_+) \arrow[d, "\prod \mathrm{trf}_\omega^\lambda"] \\
\Gamma^+\Sigma^\infty(\mathrm{E}G \times_K \omega_+) \arrow[r, "\simeq"] \arrow[d, "\Gamma^+\Sigma^\infty \kappa_\omega"] & \displaystyle\prod_{\substack{\lambda \in G^{\mathrm{ad}}/K \\ \lambda \subseteq \omega}} \Gamma^+\Sigma^\infty(\mathrm{E}G \times_K \lambda_+) \arrow[d, "\mathrm{proj}"] \\
\Gamma^+\Sigma^\infty(\mathrm{E}K \times_K K \cap \omega_+) \arrow[r, "\simeq"] & \displaystyle\prod_{\substack{\lambda \in \langle K \rangle \\ \lambda \subseteq \omega}} \Gamma^+\Sigma^\infty(\mathrm{E}G \times_K \lambda_+).
\end{tikzcd}$$

Thus for $\lambda \in \langle K \rangle$ satisfying $\lambda \subseteq \omega$, $\widetilde{\mathrm{Res}}_\omega^\lambda$ is the transfer associated with the covering (6.7), and (5.5) applied to the diagram in (6.6) gives the commutative diagram

$$\begin{tikzcd}
\Sigma^\infty(\mathrm{E}G \times_G \omega_+) \arrow[r, "\widetilde{\mathrm{Res}}_\omega^\lambda"] \arrow[d, "\simeq"] & \Gamma^+\Sigma^\infty(\mathrm{E}K \times_K \lambda_+) \arrow[d, "\simeq"] \\
\Sigma^\infty(\mathrm{E}G/C_G(x)_+) \arrow[r, "\mathrm{trf}"] & \Gamma^+\Sigma^\infty(\mathrm{E}G/C_K(x)_+).
\end{tikzcd}$$

Since $\mathrm{E}G/C_G(x)$ and $\mathrm{E}G/C_K(x)$ are models for $\mathrm{B}C_G(x)$ and $\mathrm{B}C_K(x)$, respectively, we have proved (ii). □

Acknowledgements

I want to thank Professor Ib Madsen at Aarhus University for suggesting this problem to me as well as for many enlightening discussions.

References

[1] J.F. Adams, Infinite Loop Spaces, Princeton University Press, Princeton, 1978.

[2] J.F. Adams, Stable Homotopy and Generalized Homology, The University of Chicago Press, Chicago, 1974.

[3] M. Barratt, P. Eccles, Γ^+-structures-I: A free group functor for stable homotopy theory, Topology 13 (1974) 25–45.

[4] S. Bentzen, I. Madsen, Trace maps in algebraic K-theory and the Coates–Wiles homomorphism, J. Reine Angew. Math. 411 (1990) 171–195.

[5] M. Bökstedt, Topological hochschild homology, Preprint, Bielefeld, 1985.

[6] M. Bökstedt, W.C. Hsiang, I. Madsen, The cyclotomic trace map and algebraic K-theory of spaces, Invent. Math. 111 (1993) 465–540.

[7] K.S. Brown, Cohomology of Groups, Springer, Berlin, 1982.

[8] B.I. Dundas, R. McCarthy, Stable K-theory and topological Hochschild homology, Ann. Math. 140 (1994) 685–701.
[9] L. Hesselholt, I. Madsen, On the K-theory of finite algebras over Witt vectors of finite fields, Topology 36 (1) (1997) 29–101.
[10] K. Lamotke, Semisimpliziale Algebraische Topologie, Springer, Berlin, 1968.
[11] A. Lindenstrauss, I. Madsen, Topological hochschild homology of number rings, Preprint, Aarhus University, 1996.
[12] J. Loday, Cyclic Homology, Springer, Berlin, 1992.
[13] I. Madsen, Algebraic K-Theory and Traces, Current Developments in Mathematics, International Press, 1995.
[14] J.P. May, The Geometry of Iterated Loop Spaces, Springer, Berlin, 1972.
[15] J.P. May, E_∞ spaces, group completions, and permutative categories, London Math. Soc. Lecture Note Ser. 11 (1974) 61–94.
[16] C. Schlichtkrull, On the restriction map in topological cyclic homology, Preprint, Aarhus, 1996.
[17] G. Segal, Categories and cohomology theories, Topology (1974) 293–312.
[18] F. Waldhausen, Algebraic K-theory of topological spaces. II, Algebraic Topology, Aarhus, Lecture Notes in Mathematics, vol. 763, Springer, Berlin, 1978, pp. 356–394.